STUDENT'S NAME: ______________________________

TEACHER'S NAME: ______________________________

2018	HIGHER SCHOOL CERTIFICATE **EXAMINATION**

Mathematics Extension 2

Assessment Task 4

Trial Examination

Examiners~ G. Huxley, G.Rawson, P. Biczo

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided.
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- This test paper must **NOT** be removed from the examination

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 20 minutes for this section

Section II – 90 marks (pages 5–14)

- Attempt Questions 11–16
- Allow about 1 hour 40 minutes for this section

Section 1
10 marks
Attempt Questions 1 – 10 Allow about 20 minutes for this section.
Use the multiple choice answer sheet provided for Questions 1 – 10

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1. Which of the following is the solution to the quadratic equation:

$$ix^2 + x + 2i = 0?$$

A $-i,\ -2i$ B $-i,\ 2i$ C $i,\ -2i$ D $i,\ 2i$

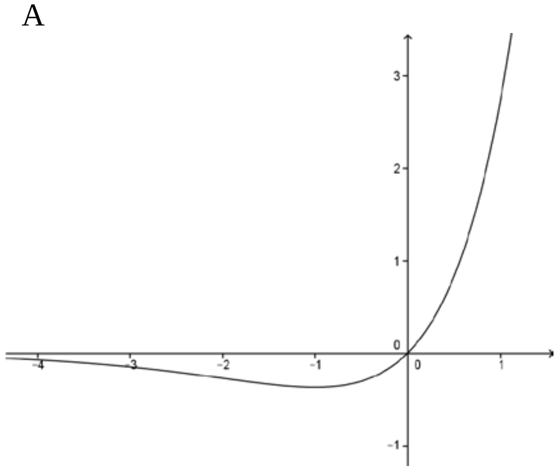
2. The circle $|z-3-2i| = 2$ is intersected exactly twice by which of the following lines?

A $|z-3-2i| = |z-5|$ B $|z-i| = |z+1|$

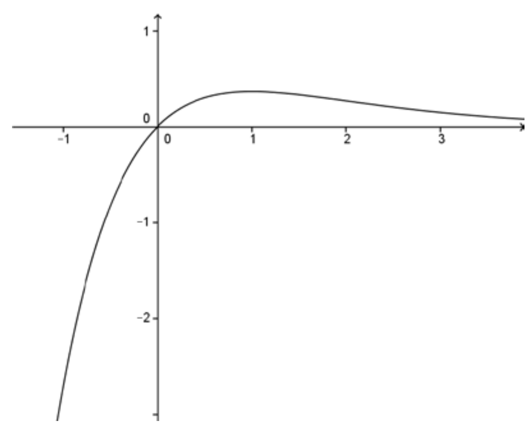
C Re(z) = 5 D Im(z) = 0

3. Which of the following is the graph of $y = xe^{-x}$?

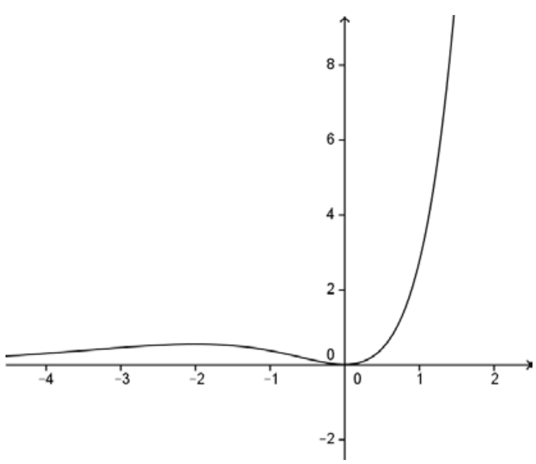
A



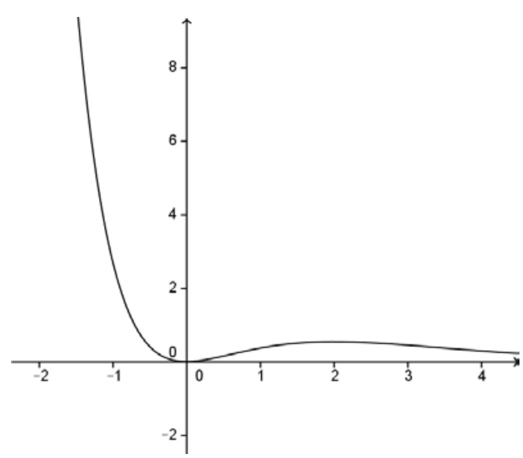
B



C



D



4. A hyperbola has its centre at the origin and its foci on the *x*-axis.
The distance between the foci is 16 units and the distance between the directrices is 4 units.
What is the equation of the hyperbola?

A $\frac{x^2}{48}-\frac{y^2}{16}=1$ B $\frac{x^2}{16}-\frac{y^2}{48}=1$

C $\frac{x^2}{4}-\frac{y^2}{16}=1$ D $\frac{x^2}{16}-\frac{y^2}{4}=1$

5. The polynomial $P(x)=x^3-5x^2-8x+48$ has double integer root at $x=\alpha$.
What is the value of α ?

A $\alpha=-3$ B $\alpha=0$

C $\alpha=3$ D $\alpha=4$

6. Let α, β and γ be the roots of the equation $x^3+3x^2+4=0$.
Which of the following polynomial equations has roots α^2, β^2 and γ^2?

A $x^3-9x^2-24x-4=0$ B $x^3-9x^2-12x-4=0$

C $x^3-9x^2-24x-16=0$ D $x^3-9x^2-12x-16=0$

7. A committee of 5 people is to be chosen from a group of 6 girls and 4 boys.
How many different committees could be formed that have at least one boy?

A $\binom{10}{5}-1$ B $\binom{4}{1}+\binom{6}{4}$

C $\binom{4}{1}\times\binom{6}{4}$ D $\binom{10}{5}-6$

8. What is the solution to the equation $\frac{x(5-x)}{x-4}\geq -3$?

A $2\leq x<4$ or $x\geq 6$ B $1\leq x<4$ or $x\geq 5$

C $4<x\leq 6$ or $x\leq 2$ D $4>x\leq 5$ or $x\leq 1$

9. Which of the following is an expression for $\int xe^{\frac{x}{2}}\,dx$?

A $\frac{1}{2}xe^{\frac{x}{2}}-\frac{1}{4}e^{\frac{x}{2}}+c$

B $\frac{1}{2}xe^{\frac{x}{2}}-\frac{1}{2}e^{\frac{x}{2}}+c$

C $2xe^{\frac{x}{2}}-2e^{\frac{x}{2}}+c$

D $2xe^{\frac{x}{2}}-4e^{\frac{x}{2}}+c$

10. The region bounded by $y=x^2$, the x-axis and the line $x = 2$ is rotated about the line $x = 2$. Using the method of circular discs to calculate the volume generated, which of the following gives the area of the circular cross-section of each disc?

A πy^2

B πx^2

C $2\pi-2\sqrt{y}$

D $\pi\left(4-4\sqrt{y}+y\right)$

Section 2
90 marks
Attempt Questions 11 – 16 **Allow about 2 hours and 40 minutes for this section**
Answer each question in a separate answer booklet.
All necessary working should be shown in every question.

Question 11 (15 marks) Start a new answer booklet. **Marks**

(a) (i) Express $1-\sqrt{3}\,i$ in modulus - argument form. **2**

(ii) Hence, evaluate $\left(1-\sqrt{3}\,i\right)^4$, writing your answer in the form $a+b\,i$, where *a* and *b* are real. **2**

(b) On the **same** Argand diagram, shade the region simultaneously defined by

$$0 \le \arg(z) \le \frac{\pi}{3}$$

$$|z| \le 2$$

$$\text{Im}(z) \le 1$$ **2**

(c) (i) Determine all five solutions to the equation

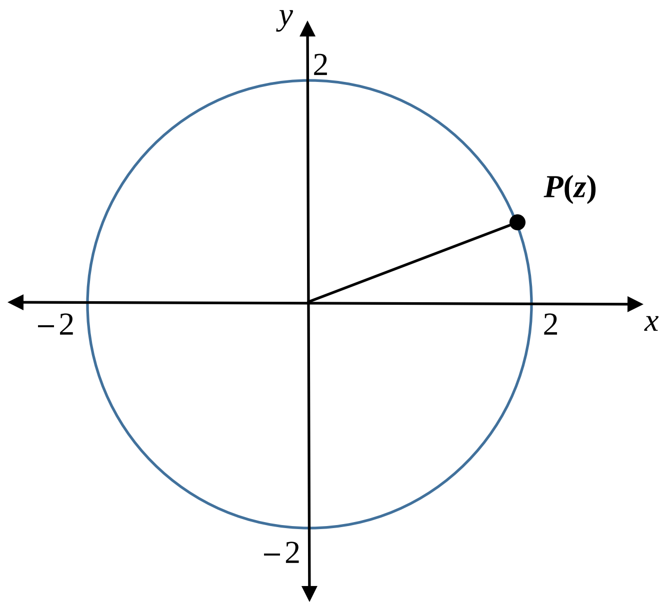
$$z^5 = -1$$

Show these on the Argand diagram. **2**

(ii) Hence, or otherwise, show that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$. **1**

Question 11 continues on the next page

(d)



Carefully copy the diagram above into your answer booklet, showing the point P representing the complex number z on the Argand diagram.

Mark the following points on your diagram:

(i) Q, representing the complex number $\dfrac{1}{z}$ **1**

(ii) R, representing the complex number $z-\dfrac{i}{z}$ **1**

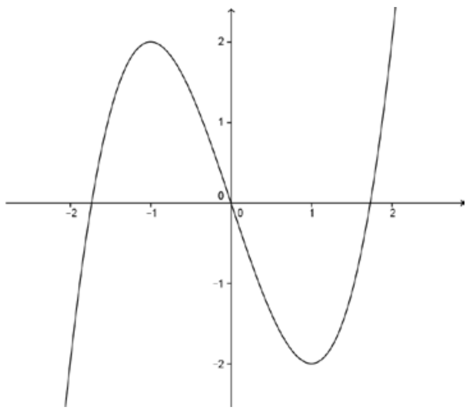
(e) It is given that $z=\dfrac{1}{1+\cos\theta+i\sin\theta}$ where $0\le\theta\le\dfrac{\pi}{2}$.

(i) Show that $z=\dfrac{1}{2}-i\dfrac{1}{2}\tan\dfrac{\theta}{2}$. **2**

(ii) Hence find $|z|$ and $\arg(z)$. **2**

Question 12 (15 marks) Start a new answer booklet. **Marks**

(a)



The curve shown has equation $f(x) = x^3 - 3x$.

Copy the curve into your answer booklet. Your sketch should be approximately $\frac{1}{3}$ of a page.

(i) On the same diagram, sketch the curve $y^2 = x^3 - 3x$.
Your diagram must clearly indicate any points of intersection with the curve $y = x^3 - 3x$. **2**

(ii) Sketch the graph of $y = \left| f\left(|x|\right) \right|$. **2**

(b) Consider the curve $f(x) = \cos^{-1}\left(e^x\right)$.

State the domain and range of the function. **2**

Question 12 continues on the next page

(c) Consider the curve $f\left(x\right)=\dfrac{e^x-1}{e^x+1}$.

(i) Show that $y=f\left(x\right)$ is an odd function. **1**

(ii) Show that $y=f\left(x\right)$ is an increasing function. **1**

(iii) Sketch the curve $y=f\left(x\right)$, showing clearly any intercepts with the coordinate axes or the equations of any asymptotes. **2**

(iv) Find the values of k for which the equation $\dfrac{e^x-1}{e^x+1}=kx$ has 3 real solutions. **2**

(v) Sketch the graph of $y=\dfrac{e^x+1}{e^x-1}$. **2**

(vi) Find the equation of the inverse function $y=f^{-1}\left(x\right)$. **1**

Question 13 (15 marks) Start a new answer booklet. **Marks**

(a) Consider the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

(i) Calculate the eccentricity and hence find the foci and the directrices of the ellipse **3**

(ii) Derive the equation of the tangent at $P\left(4\cos\theta,3\sin\theta\right)$. **2**

(iii) Show that the tangent at P cuts the positive directrix at $M\left(\frac{16\sqrt{7}}{7},\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}\right)$ **2**

(b) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S and S'.

It is given that the normal at P has equation:

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

The normal at P meets SS' at G.

(i) Show that:

$$PG^2=a^2\left(1-e^2\right)\left(1-e^2\cos^2\theta\right)$$

and that:

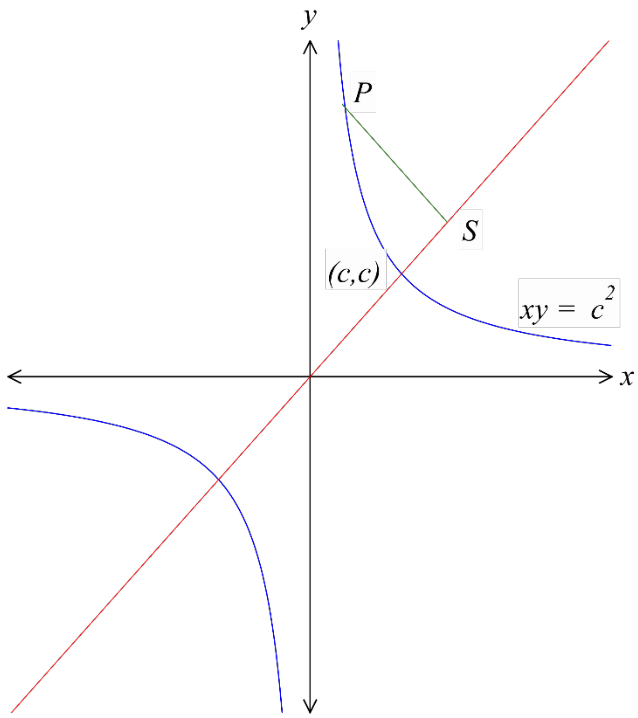
$$PS^2=a^2\left(1-e\cos\theta\right)^2$$

2

(ii) Hence show that: $PG^2=\left(1-e^2\right)PS.PS'$ **2**

Question 13 continues on the next page.

(c) The point $P\left(cp, \dfrac{c}{p}\right)$ with $p > 0$ lies on the rectangular hyperbola $xy = c^2$ with focus S. The point T divides the interval PS in the ratio 1:2.



(i) Determine the coordinates of T. **2**

(ii) Find the equation of the locus of T as P moves on the hyperbola **2**

Question 14 (15 marks) Start a new answer booklet. **Marks**

(a) Define $P(x)=x^3-2x^2+3x+2$, and let α, β and γ be the roots of $P(x)=0$.

(i) By considering $\alpha^2+\beta^2+\gamma^2$, explain why $P(x)=0$ has only one real solution **1**

(ii) Find the monic polynomial having roots $\alpha\beta, \alpha\gamma, \beta\gamma$. **3**

(b) A sequence is defined by:
$a_1=5$, $a_2=13$ **and** $a_{n+2}=5a_{n+1}-6a_n$ for all natural numbers *n*.

Show by Mathematical Induction that $a_n=2^n+3^n$. **3**

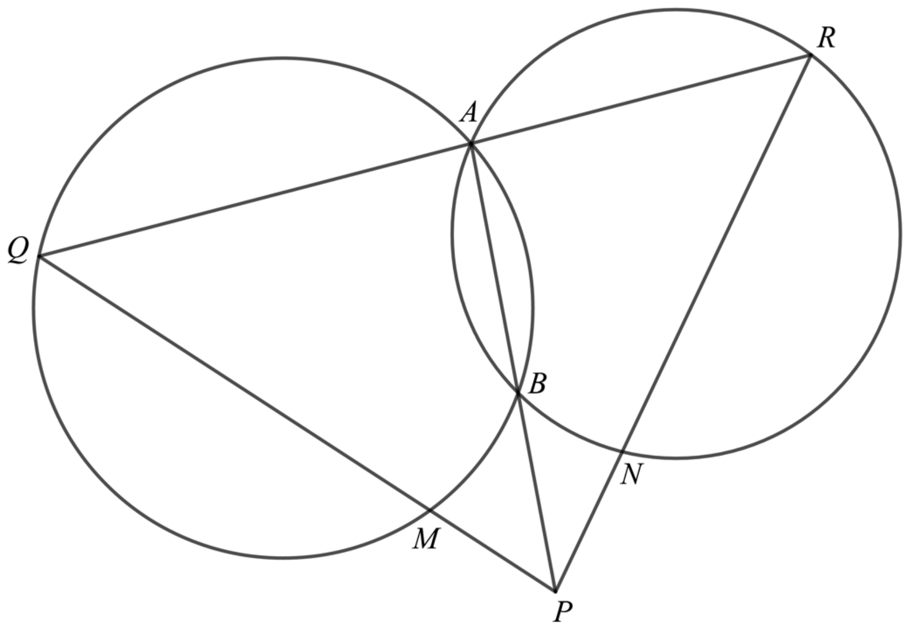
(c) (i) Show that $a^2+9b^2\geq 6ab$, where *a* and *b* are real numbers. **1**

(ii) Hence, or otherwise, show that $a^2+5b^2+9c^2\geq 3(ab+bc+ac)$. **2**

(iii) Hence if $a>b>c>0$, show that $a^2+5b^2+9c^2>9bc$. **1**

Question 14 continues on the next page.

(d) In the diagram below, two circles intersect at A and B. Chord QA on one circle is produced to cut the other circle at R. From P, on AB produced, secants are drawn to Q and R, cutting the circles at M and N respectively.



(i) Show that $PMBN$ is a cyclic quadrilateral **2**

(ii) Hence, or otherwise, show that $MQRN$ is a cyclic quadrilateral **2**

Question 15 (15 marks) Start a new answer booklet. **Marks**

(a) Find $\int \frac{\cos\theta}{\sin^5\theta} d\theta$ **2**

(b) Find $\int \frac{dx}{x^2+2x+2}$ **2**

(c) (i) Find real numbers *a*, *b* and *c* such that $\frac{9-x}{(1+x^2)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1+x}$ **2**

(ii) Hence, or otherwise, find $\int \frac{9-x}{(1+x^2)(1+x)} dx$ **3**

(d) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ where $n \geq 2$

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ **3**

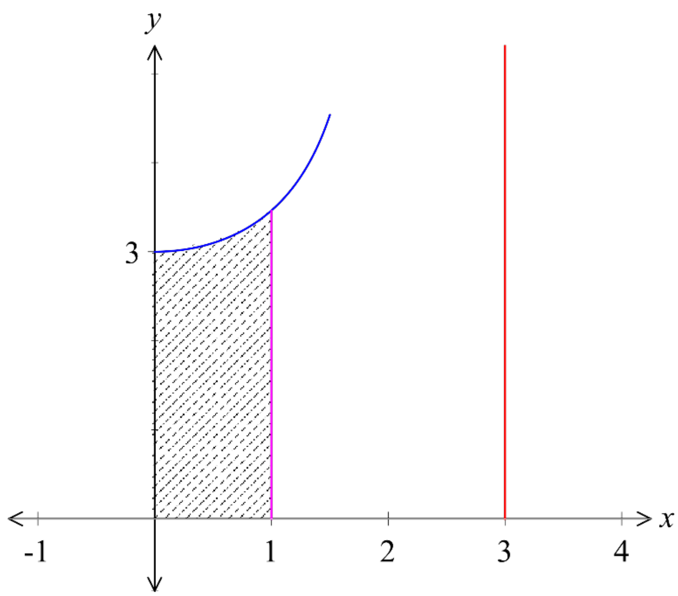
(ii) Hence, or otherwise, evaluate $\int_0^2 \left(4-x^2\right)^{\frac{5}{2}} dx$ **3**

Question 16 (15 marks) Start a new answer booklet. **Marks**

(a) The region enclosed by the curves $y = \sqrt{x}$ and $y = x$ between $x = 0$ and $x = 1$ is rotated about the x-axis to form a solid. Use the method of slices to obtain the volume of this solid. **4**

(b) A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$ and the line $x = a$, about the x-axis
Determine the volume of this solid **3**

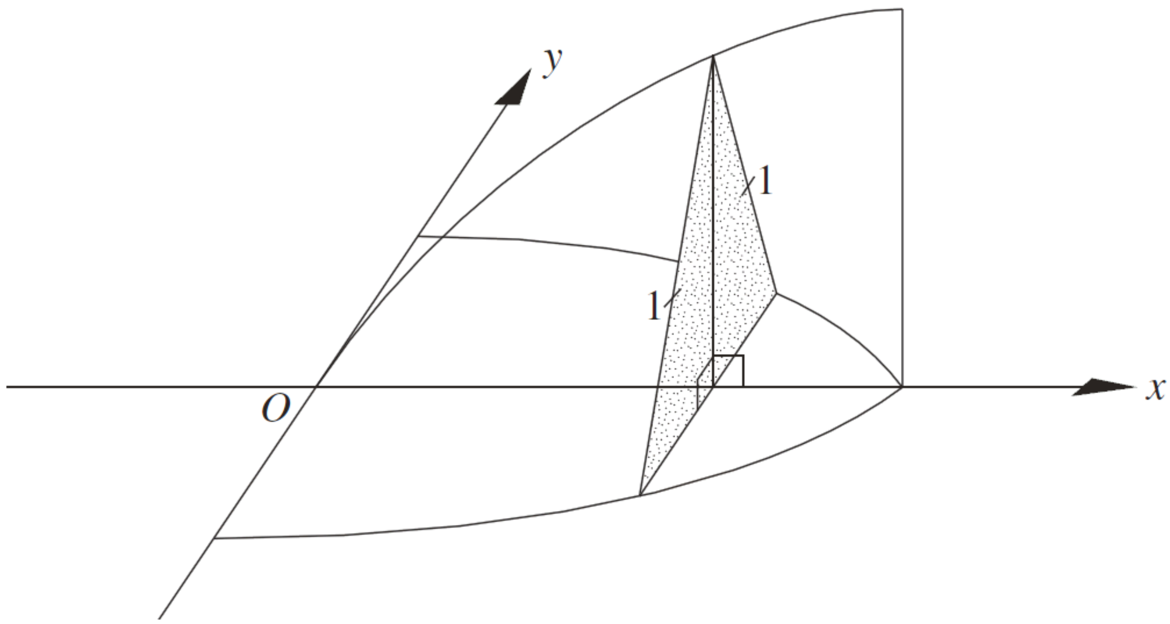
(c) The region between the curve $y = \dfrac{6}{\sqrt{4 - x^2}}$, the x-axis, $x = 0$ and $x = 1$,
is rotated about the line $x = 3$.



Give an expression for the integral that will give the volume of the solid that is generated.
You DO NOT need to evaluate the integral. **4**

(d) The base of a solid is formed by the area bounded by $y = \cos x$ and $y = -\cos x$ for $0 \le x \le \frac{\pi}{2}$.

Vertical cross-sections of the solid taken parallel to the y-axis are in the shape of isosceles triangles with the equal sides of length 1(one) unit as shown in the diagram.



Find the volume of the solid. **4**

END OF EXAMINATION

Year 12 2018 Extension 2 Trial Examination Solutions.

Multiple Choice:	**1. B**	**2. A**	**3. B**	**4. B**	**5. D**
	6. C	**7. D**	**8. C**	**9. D**	**10. D**

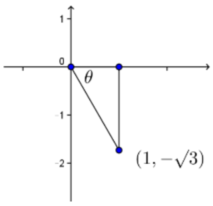
Year 12 Higher School Certificate — Mathematics Extension 2 — Task 4 2018
Question No. 11 — Solutions and Marking Guidelines

Outcomes Addressed in this Question

E3 **uses the relationship between algebraic and geometric representations of complex numbers** and of conic sections

<table>
<tr><th>Outcome</th><th>Solutions</th><th>Marking Guidelines</th></tr>
<tr><td>E3</td><td>

(a) (i) $\left|1-\sqrt{3}\,i\right| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$



$\tan\theta = \sqrt{3}$

$\therefore \arg\left(1-\sqrt{3}i\right) = -\dfrac{\pi}{3}$

$\therefore 1-\sqrt{3}i = 2\left(\cos\left(-\dfrac{\pi}{3}\right)+i\sin\left(-\dfrac{\pi}{3}\right)\right)$

</td><td>2 marks : correct solution

1 mark : significant progress towards correct solution</td></tr>
<tr><td>E3</td><td>

(ii) $\left(1-\sqrt{3}i\right)^4 = \left[2\left(\cos\left(-\dfrac{\pi}{3}\right)+i\sin\left(-\dfrac{\pi}{3}\right)\right)\right]^4$

$= 2^4\left(\cos\left(-\dfrac{4\pi}{3}\right)+i\sin\left(-\dfrac{4\pi}{3}\right)\right)$

$= 16\left(\cos\left(\dfrac{2\pi}{3}\right)+i\sin\left(\dfrac{2\pi}{3}\right)\right)$

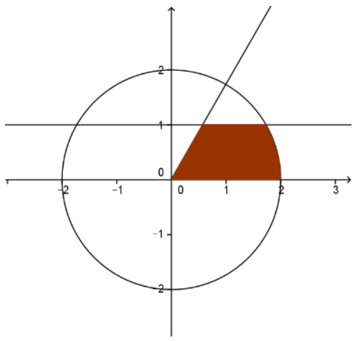
$= 16\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)$

$= -8+8\sqrt{3}i$

</td><td>2 marks : correct solution

1 mark : significant progress towards correct solution</td></tr>
<tr><td>E3</td><td>

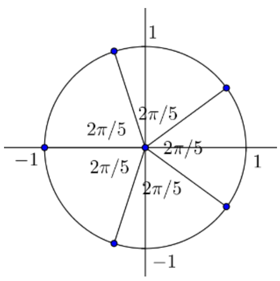
(b)



</td><td>2 marks: correct solution

1 marks: substantial progress towards solution</td></tr>
<tr><td>E3</td><td>

(c) The complex roots of -1 are evenly spaced around the unit circle, with a root at -1 and $\dfrac{2\pi}{5}$ radians apart.



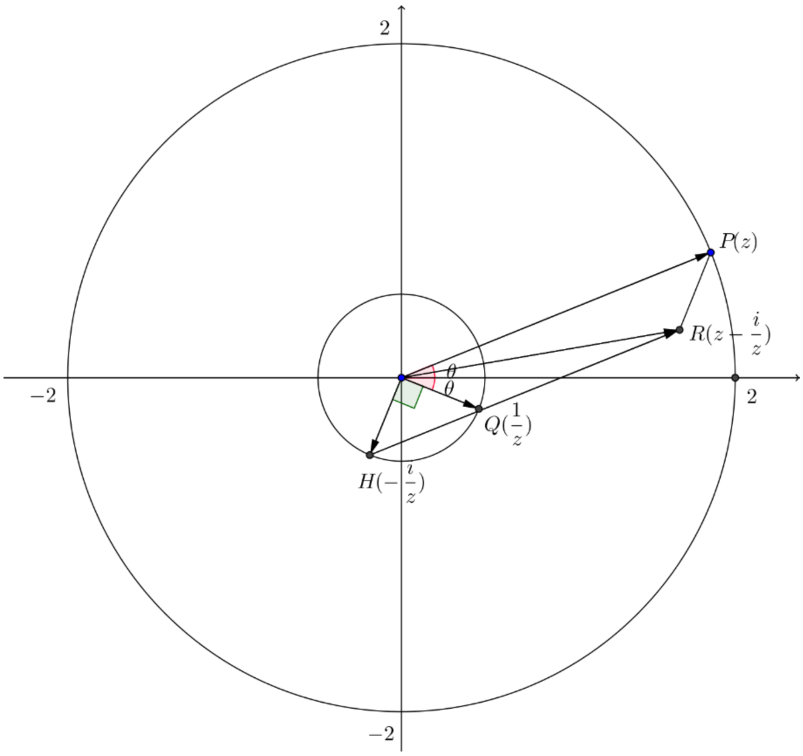
The solutions to $z^5 = -1$ are

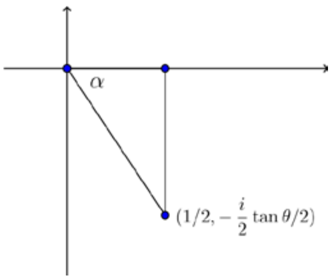
$z = -1,\quad \cos\dfrac{\pi}{5}+i\sin\dfrac{\pi}{5},\quad \cos\dfrac{3\pi}{5}+i\sin\dfrac{3\pi}{5},$

$\cos\left(-\dfrac{\pi}{5}\right)+i\sin\left(-\dfrac{\pi}{5}\right),\quad \cos\left(-\dfrac{3\pi}{5}\right)+i\sin\left(-\dfrac{3\pi}{5}\right)$

</td><td>2 marks: correct solution

1 mark: substantial progress towards solution</td></tr>
</table>

E3	(d) (i) $\frac{1}{z} = z^{-1} = \left(2\left(\cos\theta + i\sin\theta\right)\right)^{-1}$ $= \frac{1}{2}\left(\cos\left(-\theta\right) + i\sin\left(-\theta\right)\right)$	1 mark : correct solution
E3	(ii) As $-\frac{i}{z} = -i \times \frac{1}{z}$, $H(-\frac{i}{z})$ is the point representing $\frac{1}{z}$ rotated $90°$ clockwise. As $z - \frac{i}{z} = z + \left(-\frac{i}{z}\right)$ can use parallelogram rule.	2 marks : correct solution 1 mark: substantial progress towards solution
E3		
E3	(e) $z = \frac{1}{1 + \cos\theta + i\sin\theta}$ $= \frac{1}{1 + \frac{1-t^2}{1+t^2} + \frac{2it}{1+t^2}}$, where $t = \tan\frac{\theta}{2}$ $= \frac{1+t^2}{1+t^2+1-t^2+2it}$ $= \frac{1+t^2}{2+2it}$ $= \frac{\left(1+it\right)\left(1-it\right)}{2\left(1+it\right)}$ $= \frac{1-it}{2}$ $\therefore z = \frac{1}{2} - i\frac{1}{2}\tan\frac{\theta}{2}.$ (ii) $\lvert z\rvert = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\tan\frac{\theta}{2}\right)^2}$	2 marks: correct solution 1 mark: substantial progress towards solution

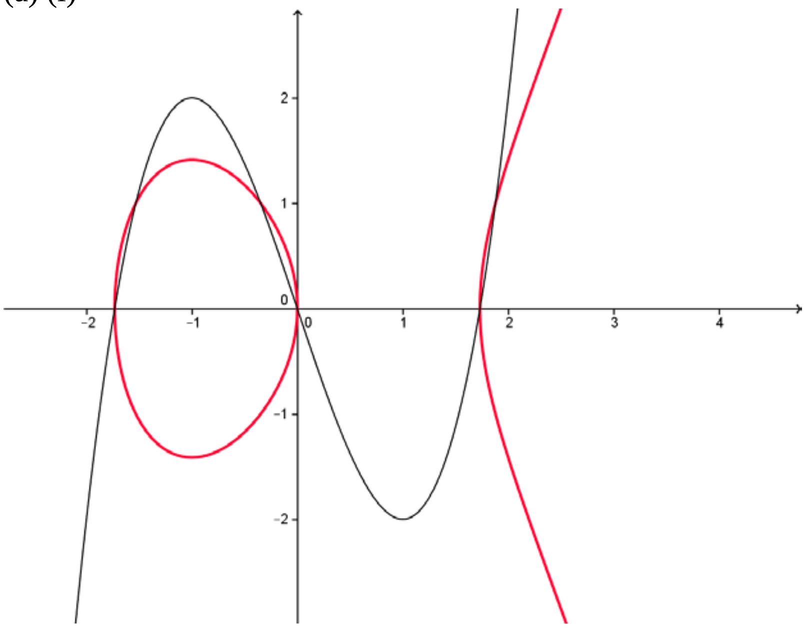
E3	$= \sqrt{\frac{1}{4} + \frac{1}{4}\tan^2\frac{\theta}{2}}$ $= \sqrt{\frac{1}{4}\left(1 + \tan^2\frac{\theta}{2}\right)}$ $= \frac{1}{2}\sqrt{\sec^2\frac{\theta}{2}}$ $\therefore \lvert z\rvert = \frac{1}{2}\sec\frac{\theta}{2}$ as $0 \le \theta \le \frac{\pi}{2}$. From the diagram, $\tan\alpha = \dfrac{\frac{1}{2}\tan\frac{\theta}{2}}{\frac{1}{2}}$ $\tan\alpha = \tan\frac{\theta}{2} \quad \alpha = \frac{\theta}{2}$ $\therefore \arg(z) = -\frac{\theta}{2}$  Note: for complex number questions, it is often necessary to draw a circle – this is best done using a template or compass. Having a protractor is also an advantage for questions like (c) and (d). In these questions many students did not indicate important properties, which resulted in marks not being able to be awarded, such as: (c) the roots lie on a unit circle and are equally spaced (d), (i) *Q* lies on a circle of radius ½; arg(*P*) = – arg(*Q*) should be indicated by showing equal sized angles (ii) how *R* was located – rotations of 90° and addition/ subtraction of vectors by drawing parallelograms should be indicated to demonstrate your method	2 marks : both modulus and argument correct 1 mark: one of above correct

Outcomes Addressed in this Question

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

<table>
<tr><th>Outcome</th><th>Solutions</th><th>Marking Guidelines</th></tr>
<tr><td>E6</td><td>

(a) (i)



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2 marks : correct solution

1 mark : significant progress towards correct solution

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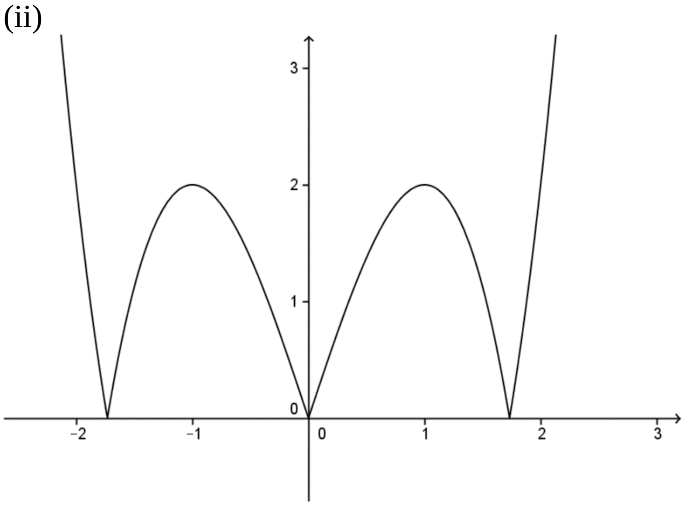
<tr><td>E6</td><td>

(ii)



</td><td>

2 marks : correct solution

1 mark : significant progress towards correct solution

</td></tr>
<tr><td>E6</td><td>

(b) Can find $\cos^{-1}(g(x))$ provided $-1 \le g(x) \le 1$. Since $e^x > 0$, need $e^x \le 1$. This occurs when $x \le 0$,

$\therefore$ domain is $x \le 0$ (where x is a real number).

When $x \le 0$, $0 < e^x \le 1$.

Cos inverse of values between 0 and 1 (including 1) gives answers between 0 and $\frac{\pi}{2}$ (but not including $\frac{\pi}{2}$).

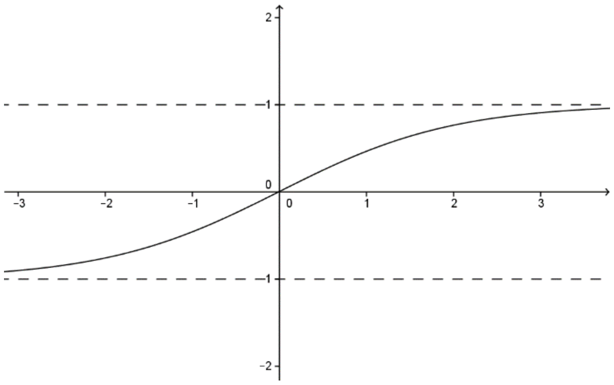
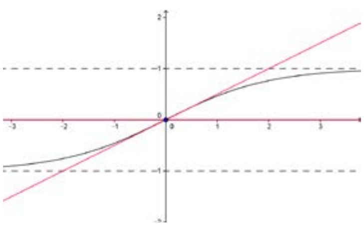
$\therefore$ range is $0 \le y < \frac{\pi}{2}$ (where y is a real number).

</td><td>

2 marks: correct domain and correct range

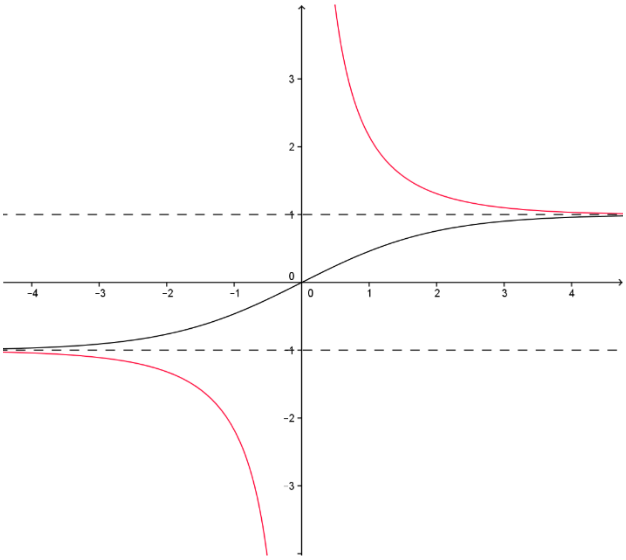
1 marks: one correct of above

</td></tr>
</table>

E6	(c) (i) $f(x)=\dfrac{e^x-1}{e^x+1}$ $\therefore f(-x)=\dfrac{e^{-x}-1}{e^{-x}+1}$ $=\dfrac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1}$ $=\dfrac{1-e^x}{1+e^x}$ $=\dfrac{-(e^x-1)}{e^x+1}$ $=-f(x)$ $\therefore$ odd function	1 mark: correct solution
E6	(ii) $f(x)=\dfrac{e^x-1}{e^x+1}$ $f'(x)=\dfrac{(e^x+1)e^x-(e^x-1)e^x}{(e^x+1)^2}$ $f'(x)=\dfrac{2e^x}{(e^x+1)^2}$ As $e^x>0$ for all values of x and the denominator is positive, $f'(x)>0$. $\therefore y = f(x)$ is an increasing function	1 mark: correct solution
E6	(iii)  	2 marks : correct solution 1 mark: substantial progress towards solution
E6	(iv) $\dfrac{e^x-1}{e^x+1}=kx$ has 3 solutions when the line $y=kx$ intersects the curve $y=\dfrac{e^x-1}{e^x+1}$ in three places. This will occur when the gradient is positive, and the gradient is less than the gradient of the tangent.  Tangent to the curve at (0, 0) has gradient $f'(0)=\dfrac{2e^0}{(e^0+1)^2}=\dfrac{1}{2}$ As the line $y=kx$ has gradient k, $0<k<\dfrac{1}{2}$	2 marks: correct solution 1 mark: substantial progress towards solution

E6

(v) Since $\frac{e^x+1}{e^x-1}$ is the reciprocal of $\frac{e^x-1}{e^x+1}$, need to draw $y=\frac{1}{f(x)}$. Graph drawn in red is $y=\frac{e^x+1}{e^x-1}$



2 marks : correct solution

1 mark : significant progress towards correct solution

E6

(vi) Inverse function is $x=\frac{e^y-1}{e^y+1}$,

$$xe^y+x=e^y-1$$
$$x+1=e^y-xe^y$$
$$e^y=\frac{x+1}{1-x}$$
$$\therefore y=\ln\left(\frac{x+1}{1-x}\right)$$

1 marks : correct solution

Note:

- More care needs to be taken and more attention paid to detail in graphs such as (a)(i) & (ii).
  (a)(i) stated that points of intersection with the original curve (when y = 1) needed to be clearly indicated and which curve was above/below. Using different colours is a good way to distinguish between different graphs. If students did not mark that the graphs intersected at y = 1, marks could not be awarded. Also, any vertical tangents should appear as vertical tangents; lines where $y\to\infty$ should not appear to be horizontal or approaching a limit.
  For a graph out of 2 marks, such as (a)(ii) scale should be indicated on the axes; if a graph is symmetrical the graph should reflect this.

Year 12 (2018)	Mathematics Extension 2	AT4 2018 HSC
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Part / Outcome	Solutions	Marking Guidelines
(a)	(i) $\frac{x^2}{16}+\frac{y^2}{9}=1 \quad \rightarrow a=4, b=3$ $e=\frac{\sqrt{7}}{4} \quad$ foci $=\left(\pm\sqrt{7},0\right)$ directrices: $x=\pm\frac{16}{\sqrt{7}}$	**(a)(i) 3marks:** All 3 components correct, 1 mark per component.
	(ii) $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\frac{3\cos\theta}{4\sin\theta}$ Tangent: $y-3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}\left(x-4\cos\theta\right)$ $\rightarrow 4y\sin\theta+3x\cos\theta=12$ $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$	**(ii) 2 marks:** Correct solution with working. **1 mark:** A correct substitution of all information.
	(iii) Directrix: $x=\frac{16}{\sqrt{7}}=\frac{16\sqrt{7}}{7}$ Sub into tangent equation: $\frac{16\sqrt{7}\cos\theta}{28}+\frac{y\sin\theta}{3}=1$ $\frac{y\sin\theta}{3}=\frac{28-16\sqrt{7}\cos\theta}{28}$ $y=\frac{84-48\sqrt{7}\cos\theta}{28\sin\theta}$ $y=\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}$ As required.	**(iii) 2 marks:** Correct solution with working. **1 mark:** A correct substitution of all information.

(b)	(i) At $G, y=-0$ $\therefore \frac{ax}{\cos\theta}=a^2-b^2=a^2e^2$ since it's an ellipse $\therefore x=ae^2\cos\theta$ $PG^2=\left(a\cos\theta-ae^2\cos\theta\right)^2+\left(b\sin\theta-0\right)^2$ $=a^2\left(\cos^2\theta\left(1-e^2\right)^2+\frac{b^2}{a^2}\sin^2\theta\right)$ $=a^2\left(\cos^2\theta\left(1-e^2\right)^2+\left(1-e^2\right)\sin^2\theta\right)$ $=a^2\left(1-e^2\right)\left(\cos^2\theta-e^2\cos^2\theta+\sin^2\theta\right)$ $=a^2\left(1-e^2\right)\left(1-e^2\cos^2\theta\right)$ $S=\left(ae,0\right)$ $PS^2=\left(a\cos\theta-ae\right)^2+\left(b\sin\theta-0\right)^2$ $=a^2\left(\cos^2\theta-2e\cos\theta+e^2+\left(1-e^2\right)\sin^2\theta\right)$ $=a^2\left(\cos^2\theta-2e\cos\theta+e^2+\sin^2\theta-e^2\left(1-\cos^2\theta\right)\right)$ $=a^2\left(1-2e\cos\theta+e^2\cos^2\theta\right)$ $=a^2\left(1-e\cos\theta\right)^2$ (ii) From (i) $PS=a\left(1-e\cos\theta\right)$ Hence $PS'=a\left(1+e\cos\theta\right)$ $RHS=\left(1-e^2\right)PS.PS'$ $=\left(1-e^2\right)\left(\left(a\right)\left(1-e\cos\theta\right)\right)\left(\left(a\right)\left(1+e\cos\theta\right)\right)$ $=a^2\left(1-e^2\right)\left(1-e^2\cos^2\theta\right)$ $=PG^2=LHS$	**(b) (i) 2 marks:** Correct solutions **1 mark:** 1 of the solutions correct. *Note: There was a lot of working required per mark, and it seemed that many students used way too much time on this question. If the first attempt didn't work, it was worth considering moving on and coming back to this question later..* **(ii) 2 marks:** Correct solution in required form. **1 mark:** Relevant progress.
(c)	(i) $P=\left(cp,\frac{c}{p}\right)$ $S=\left(c\sqrt{2},c\sqrt{2}\right)$ $T=\left(\frac{2cp+c\sqrt{2}}{3},\frac{2c+cp\sqrt{2}}{3p}\right)$ (ii) From T: $p=\frac{3x-c\sqrt{2}}{2c}$ and $p=\frac{2c}{3y-c\sqrt{2}}$ $\therefore \frac{3x-c\sqrt{2}}{2c}=\frac{2c}{3y-c\sqrt{2}}$ $\left(3x-c\sqrt{2}\right)\left(3y-c\sqrt{2}\right)=4c^2$ Which is a rectangular hyperbola with asymptotes : $x=\frac{c\sqrt{2}}{3}$, $y=\frac{c\sqrt{2}}{3}$	**(c) (i) 2 marks:** Both answers correct. **1 mark:** One of the answers correct, or both answers incorrect from only 1 error. **(ii) 2 marks:** Solution in a form that can be interpreted as a hyperbola. *Just substituting for p into one of the x or y values did not gain full marks.* **1 mark:** A single substitution for p in an attempt to eliminate the parameter.

Outcomes Addressed in this Question

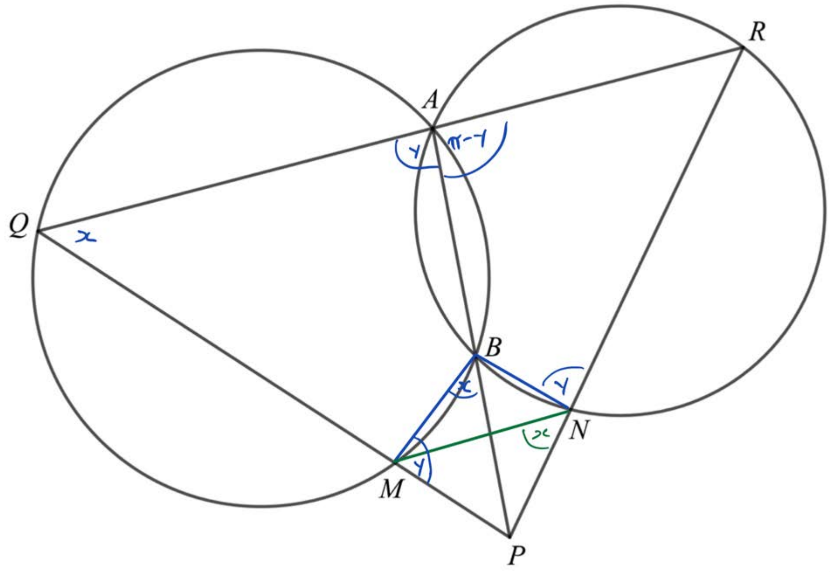
E2 - chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E4 - uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials

E9 - communicates abstract ideas and relationships using appropriate notation and logical argument

Outcome	Solutions	Marking Guidelines
E4, E9	**(a) (i)** $P(x)=x^3-2x^2+3x+2$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $=2^2-2(3)$ $=-2$ If α, β, γ are all real, then $\alpha^2+\beta^2+\gamma^2>0$, which is not true, so there is at least one complex root. The coefficients of $P(x)$ are all rational, so complex roots occur in conjugate roots. $\therefore$ there are two complex roots **<u>and one real root</u>**.	**<u>1 mark:</u>** correct solution. <u>Full</u> explanation needed
E4	**(ii)** $\alpha\beta\gamma=-\frac{d}{a}=-2$ roots $\alpha\beta, \alpha\gamma, \beta\gamma$ become $\alpha\beta=-\frac{2}{\gamma}, \alpha\gamma=-\frac{2}{\beta}, \beta\gamma=-\frac{2}{\alpha}$ so let $y=-\frac{2}{x} \Rightarrow x=-\frac{2}{y}$ $P(x)=x^3-2x^2+3x+2$ $P\left(-\frac{2}{y}\right)=\left(-\frac{2}{y}\right)^3-2\left(-\frac{2}{y}\right)^2+3\left(-\frac{2}{y}\right)+2$ $0=-\frac{8}{y^3}-\frac{8}{y^2}-\frac{6}{y}+2$ $0=-8-8y-6y^2+2y^3$ $P(y)=y^3-3y^2-4y-4$ (monic) *Question 14 continued…*	**<u>3 marks:</u>** correct solution (must be monic) **<u>2 mark:</u>** substantial progress towards correct solution **<u>1 mark:</u>** partial progress towards correct solution

E2, E9	(b) Let $S(n)$ be the statement $a_n = 2^n + 3^n$ step 1: Show true for $n = 1,2$ $n=1: 2^1 + 3^1 = 5,$ so $S(1)$ is true $n=2: 2^2 + 3^2 = 13,$ so $S(2)$ is true step 2: Assume $S(k), S(k+1)$ are true ie $a_k = 2^k + 3^k$ and $a_{k+1} = 2^{k+1} + 3^{k+1}$ step 3: Prove $S(k+2)$ is true ie prove $a_{k+2} = 2^{k+2} + 3^{k+2}$ $\text{LHS} = a_{k+2}$ $= 5a_{k+1} - 6a_k$ (given recursive formula) $= 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$ (by assumption) $= 5.2^{k+1} + 5.3^{k+1} - 3.2.2^k - 3.2.2^k$ $= 5.2^{k+1} + 5.3^{k+1} - 3.2^{k+1} - 2.3^{k+1}$ $= 2.2^{k+1} + 3.3^{k+1}$ $= 2^{k+2} + 3^{k+2}$ $= \text{RHS}$ Hence $S(k+2)$ is true $\therefore S(k)$ is true by Mathematical Induction	**3 marks:** correct solution **2 mark:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution
E2	(c) (i) consider $a^2 + 9b^2 - 6ab$ $= (a-3b)^2 \geq 0$ $\therefore a^2 + 9b^2 - 6ab \geq 0$ and so $a^2 + 9b^2 \geq 6ab$	**1 mark:** correct solution
	(ii) $a^2 + 9b^2 \geq 6ab$ (from (i)) so $b^2 + 9c^2 \geq 6bc$ and $a^2 + 9c^2 \geq 6ac$ adding, $2a^2 + 10b^2 + 18c^2 \geq 6(ab + bc + ac)$ $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$	**2 marks:** correct solution **1 mark:** substantial progress towards correct solution
	(iii) $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$ $> 3(bc + bc + bc)$ $\begin{pmatrix} a > c \Rightarrow ab > bc \\ a > b \Rightarrow ac > bc \end{pmatrix}$ $= 3(3bc)$ $= 9bc$ ie $a^2 + 5b^2 + 9c^2 > 9bc$ *Question 14 continued…*	**1 mark:** correct solution
E2	(d)	



2 marks: correct solution

1 mark: substantial progress towards correct solution

(i) $\angle MBP = \angle AQM = x$ (ext $\angle$ of cyclic quad $BMQA$)

$\angle PMB = \angle QAB = y$ (ext $\angle$ of cyclic quad $BMQA$)

$\angle RAP = \pi - y$ (straight $\angle QAR$)

$\angle RNB = \pi - \angle RAP$ (opposite $\angle$'s of cyclic quad $ARNB$)

$= \pi - (\pi - y)$

$= y$

$\therefore \angle RNB = \angle BMP$

$\therefore$ $PMBN$ is a cyclic quad (ext $\angle$ = opposite interior $\angle$)

2 marks: correct solution

1 mark: substantial progress towards correct solution

(ii) $\angle MBP = \angle MNP$ (angles on circumference standing on arc MP of cyclic quad $PMBN$)

$\therefore \angle RQM = \angle MNP = x$

But $\angle MNP$ is the exterior angle of quad $QRNM$

$\therefore$ $QRNM$ is a cyclic quadrilateral

Year 12	Mathematics Extension 2	Task 4 (TRIAL) 2018
Question No. 15	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E8 - applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

Outcome	Solutions	Marking Guidelines
	(a) $I=\int\frac{\cos\theta}{\sin^5\theta}d\theta$ let $u=\sin\theta$, $du=\cos\theta\,d\theta$ $=\int\frac{du}{u^5}=\int u^{-5}\,du$ $=\frac{u^{-4}}{-4}+c$ $=-\frac{1}{4\sin^4\theta}+c$	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(b) $I=\int\frac{dx}{x^2+2x+2}$ $=\int\frac{dx}{(x+1)^2+1}$ $=\tan^{-1}(x+1)+c$	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(c) (i) $\frac{9-x}{(1+x^2)(1+x)}=\frac{ax+b}{1+x^2}+\frac{c}{1+x}$ $9-x=(ax+b)(1+x)+c(1+x^2)$ let $x=-1\ \Rightarrow c=5$ $x=0\ \Rightarrow b=4$ $x=1\ \Rightarrow a=-5$ i.e. $\frac{9-x}{(1+x^2)(1+x)}=\frac{-5x+4}{1+x^2}+\frac{5}{1+x}$	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(ii) $\int\frac{9-x}{(1+x^2)(1+x)}\,dx=\int\left(\frac{-5x+4}{1+x^2}+\frac{5}{1+x}\right)dx$ $=\int\frac{4}{1+x^2}dx-\int\frac{5x}{1+x^2}dx+\int\frac{5}{x+1}dx$ $=4\tan^{-1}x-\frac{5}{2}\ln(x^2+1)+5\ln(x+1)+c$ $=4\tan^{-1}x+\frac{5}{2}\ln\left(\frac{(x+1)^2}{x^2+1}\right)+c$ *Question 15 continued…*	**<u>3 marks:</u>** correct solution **<u>2 mark:</u>** substantial progress towards correct solution **<u>1 mark:</u>** partial progress towards correct solution

	(d) (i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ $= \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \sin\theta \, d\theta$ $u = \sin^{n-1}\theta \qquad dv = \sin\theta d\theta$ $du = (n-1)\sin^{n-2}\cos\theta d\theta \qquad v = -\cos\theta$ $I_n = \left[-\cos\theta \sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$ $= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta (1-\sin^2\theta) \, d\theta$ $= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2}\theta - \sin^n\theta) \, d\theta$ $= (n-1)I_{n-2} - (n-1)I_n$ $nI_n = (n-1)I_{n-2}$ $I_n = \frac{n-1}{n} I_{n-2}$	**<u>3 marks:</u>** correct solution **<u>2 mark:</u>** substantial progress towards correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(ii) $x = 2\cos\theta \qquad$ when $x = 0, \theta = \frac{\pi}{2}$ $dx = -2\sin\theta d\theta \qquad$ when $x = 2, \theta = 0$ $\int_0^2 (4-x^2)^{\frac{5}{2}} dx = -\int_{\frac{\pi}{2}}^0 (4-4\cos^2\theta)^{\frac{5}{2}} \times 2\sin\theta \, d\theta$ $= 4^{\frac{5}{2}}.2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{5}{2}} .\sin\theta \, d\theta$ $= 64\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta$ $= 64 I_6$ $= 64 \times \frac{5}{6} I_4$ $= \frac{160}{3} \times \frac{3}{4} I_2$ $= 40 \times \frac{1}{2} I_0$ $= 20\int_0^{\frac{\pi}{2}} \sin^0\theta \, d\theta = 20\int_0^{\frac{\pi}{2}} 1 \, d\theta$ $= 20[\theta]_0^{\frac{\pi}{2}}$ $= 20\left[\frac{\pi}{2} - 0\right]$ $= 10\pi$	**<u>3 marks:</u>** correct solution **<u>2 mark:</u>** substantial progress towards correct solution **<u>1 mark:</u>** partial progress towards correct solution

Year 12 (2018)	Mathematics Extension 2	AT4 2018 HSC
Question No. 16	Solutions and Marking Guidelines	
Outcomes Addressed in this Question		
E7 uses the techniques of slicing and cylindrical shells to determine volumes		

Part / Outcome	**Solutions**	**Marking Guidelines**
(a)	From $x = 0$ to $x = 1$, $y=\sqrt{x}$ is above $y=x$. By slicing, each slice will be a "washer shape" $A(x)=\pi\left(\sqrt{x}^2-x^2\right)=\pi\left(x-x^2\right)$ $\partial V=\pi\left(x-x^2\right)\partial x$ $\rightarrow V=\pi\int_0^1 x-x^2\,dx=\frac{\pi}{6}\text{units}^3$	**(a) 4 marks:** Complete solution based upon DAVIE principles. **3 marks:** Almost all elements included. **2 marks:** Significant progress. **1 mark:** Some relevant progress.
(b)	This question is a volume of revolution; each slice is a circle. Area of slice = $\pi y^2=4ax\pi$ $\text{Volume}=\lim_{\partial x\to 0}\sum_{x=0}^{a}4ax\pi\,\partial x$ $V=4a\pi\int_0^a x\,dx=2\pi a^3\text{ units}^3$	**(b) 3 marks:** Complete solution. **2 marks:** Substantial progress. **1 mark:** Some relevant progress.
(c)	Using cylindrical shells. $r=(3-x)\qquad h=y=\frac{6}{\sqrt{4-x^2}}$ $V=\lim_{\partial x\to 0}\sum_{x=0}^{1}2\pi(3-x)\left(\frac{6}{\sqrt{4-x^2}}\right)\partial x$ $V=12\pi\int_0^1\frac{3-x}{\sqrt{4-x^2}}dx$	**(c) 4 marks:** Complete solution based upon DAVIE principles. **3 marks:** Almost all elements included. **2 marks:** Significant progress. **1 mark:** Some relevant progress
(d)	Height of triangular slice = $\sqrt{1-y^2}$ Area of triangular slice = $\frac{1}{2}bh=y\sqrt{1-y^2}$ Thickness of slice is ∂x $A(x)=\cos x\sqrt{1-\cos^2 x}=\cos x\sin x$ $=\frac{1}{2}\sin 2x$ $V=\lim_{\partial x\to 0}\sum_{x=0}^{\frac{\pi}{2}}\frac{1}{2}\sin 2x\,\partial x$ $=\int_0^{\pi/2}\frac{1}{2}\sin 2x\,dx$ $=-\frac{1}{4}\cos 2x\,\Big]_0^{\pi/2}$ $=\frac{1}{2}\text{unit}^3$	**(d) 4 marks:** Complete solution based upon DAVIE principles. **3 marks:** Almost all elements included. **2 marks:** Significant progress. **1 mark:** Some relevant progress